The Transmission Line Wave Equation

Q: *So, what functions I (z) and V (z)* ***do*** *satisfy both telegrapher's equations??*

A: To make this easier, we will combine the telegrapher equations to form **one** differential equation for ***V (z)*** and **another** for ***I(z)***.

First, take the **derivative** with respect to *z* of the **first** telegrapher equation:

$$\frac{\partial}{\partial z}\left\{\frac{\partial V(z)}{\partial z} = -(R + j\omega L)\, I(z)\right\}$$

$$= \frac{\partial^2 V(z)}{\partial z^2} = -(R + j\omega L)\frac{\partial I(z)}{\partial z}$$

Note that the **second** telegrapher equation expresses the derivative of *I*(*z*) in terms of *V*(*z*):

$$\frac{\partial I(z)}{\partial z} = -(G + j\omega C)\, V(z)$$

Combining these two equations, we get an equation involving $V(z)$ **only**:

$$\frac{\partial^2 V(z)}{\partial z^2} = (R + j\omega L)(G + j\omega C)\, V(z)$$
$$= \gamma^2 V(z)$$

where it is apparent that:

$$\gamma^2 \doteq (R + j\omega L)(G + j\omega C)$$

In a **similar** manner (i.e., begin by taking the derivative of the **second** telegrapher equation), we can derive the differential equation:

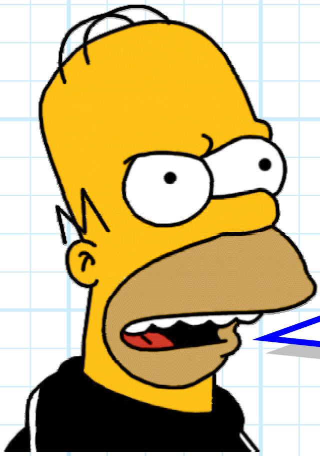
$$\frac{\partial^2 I(z)}{\partial z} = \gamma^2 I(z)$$

We have **decoupled** the telegrapher's equations, such that we now have **two** equations involving **one** function only:

$$\frac{\partial^2 V(z)}{\partial z} = \gamma^2 V(z)$$

$$\frac{\partial^2 I(z)}{\partial z} = \gamma^2 I(z)$$

Note only **special** functions satisfy these equations: if we take the double derivative of the function, the result is the **original function** (to within a constant)!



Q: *Yeah right! Every function that* ***I*** *know is* ***changed*** *after a double differentiation. What kind of "magical" function could possibly satisfy this differential equation?*

A: Such functions **do** exist !

For example, the functions $V(z) = e^{-\gamma z}$ and $V(z) = e^{+\gamma z}$ each satisfy this transmission line wave equation (**insert** these into the differential equation and see for **yourself**!).

Likewise, since the transmission line wave equation is a **linear** differential equation, a weighted **superposition** of the two solutions is **also a solution** (again, **insert** this solution to and see for **yourself**!):

$$V(z) = V_0^+ \, e^{-\gamma z} + V_0^- \, e^{+\gamma z}$$

In fact, it turns out that **any** and **all** possible solutions to the differential equations can be expressed in **this** simple form!

Therefore, the **general** solution to these wave equations (and thus the telegrapher equations) are:

$$V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{+\gamma z}$$

$$I(z) = I_0^+ e^{-\gamma z} + I_0^- e^{+\gamma z}$$

where V_0^+, V_0^-, I_0^+, I_0^-, and γ are **complex constants**.

→ It is **unfathomably** important that **you** understand what this result means!

It means that the functions $V(z)$ and $I(z)$, describing the current and voltage at **all** points z along a transmission line, can **always** be **completely** specified with just **four complex constants** (V_0^+, V_0^-, I_0^+, I_0^-)!!

We can **alternatively** write these solutions as:

$$V(z) = V^+(z) + V^-(z)$$

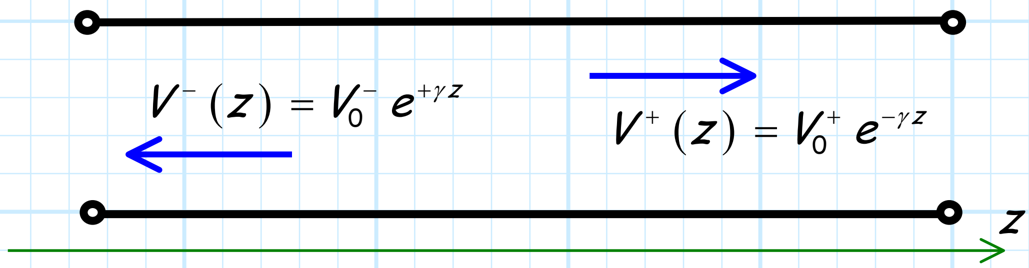
$$I(z) = I^+(z) + I^-(z)$$

where:

$$V^+(z) \doteq V_0^+ e^{-\gamma z} \qquad V^-(z) \doteq V_0^- e^{+\gamma z}$$

$$I^+(z) \doteq I_0^+ e^{-\gamma z} \qquad I^-(z) \doteq I_0^- e^{+\gamma z}$$

The two terms in each solution describe **two waves** propagating in the transmission line, **one** wave ($V^+(z)$ or $I^+(z)$) propagating in one direction (**+z**) and the **other** wave ($V^-(z)$ or $I^-(z)$) propagating in the **opposite** direction (**-z**).



Therefore, we call the differential equations introduced in this handout the **transmission line wave equations**.

Q: *So just what **are** the complex values V_0^+, V_0^-, I_0^+, I_0^- ?*

A: Consider the wave solutions at **one** specific point on the transmission line—the point $z = 0$. For example, we find that:

$$
\begin{aligned}
V^+(z=0) &= V_0^+ e^{-\gamma(z=0)} \\
&= V_0^+ e^{-(0)} \\
&= V_0^+ (1) \\
&= V_0^+
\end{aligned}
$$

In other words, V_0^+ is simply the **complex** value of the wave function $V^+(z)$ at the point $z=0$ on the transmission line!

Likewise, we find:

$$V_0^- = V^-(z=0)$$

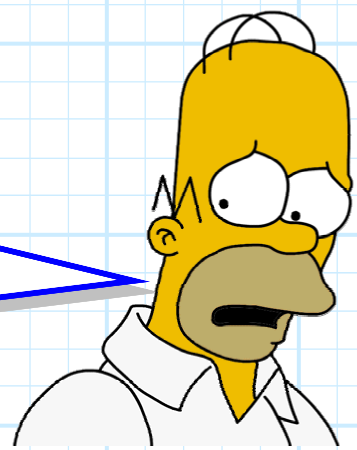
$$I_0^+ = I^+(z=0)$$

$$I_0^- = I^-(z=0)$$

Again, the four complex values V_0^+, I_0^+, V_0^-, I_0^- are **all** that is needed to determine the voltage and current at any and all points on the transmission line.

More specifically, **each** of these four complex constants completely specifies **one** of the four transmission line wave functions $V^+(z)$, $I^+(z)$, $V^-(z)$, $I^-(z)$.

Q: *But what **determines** these wave functions? How do we **find** the values of constants V_0^+, I_0^+, V_0^-, I_0^-?*



A: As you might expect, the voltage and current on a transmission line is determined by the devices **attached** to it on either end (e.g., active sources and/or passive loads)!

The precise values of V_0^+, I_0^+, V_0^-, I_0^- are therefore determined by satisfying the **boundary conditions** applied at **each end** of the transmission line—much more on this **later**!